Post 6: $\mathbb{C}-$linear categories, simplicity and semisimplicity

In this post I will follow mostly section 3 of Liang Kong and Zhi-Hao Zhang's paper "An invitation to topological orders and category theory". In particular, I will introduce another type of enriched category called $\mathbb{C}-$linear categories, as well as additional necessary concepts, such as semisimplicity. We shall see that the structure of $\mathbb{C}-$linear categories reproduces much of that observed in the previous post where we recovered linear algebra from category theory. This structure is necessary for the construction of a direct sum, from which the notions of semisimplicity and simplicity will emerge.

1 $\mathbb{C}-$linear categories

Definition ($\mathbb{C}-$linear category) A $\mathbb{C}-$linear category is a category C enriched over the category finite-dimensional vector spaces over $\mathbb{C}$, $\mathrm{Vec}_{\mathbb{C}}$; i.e. such that for any two objects $A, B \in \mathrm{ob}(\mathrm{C})$, then $\mathrm{Hom}(A, B)$ is a vector space over $\mathbb{C}$, and composition is $\mathbb{C}-$bilinear.

Definition ($0-$morphism) Let $A, B \in \mathrm{ob}(\mathrm{C})$ for C a $\mathbb{C}-$linear category. Since $\mathrm{Hom}(A, B)$ is a vector space, then there is a zero vector in it, denoted by $0 : A \to B$. This is called the $0-$morphism.

Observation Let C be a $\mathbb{C}-$linear category, and let $A, B \in \mathrm{ob}(\mathbf{C})$. $\forall C \in \mathrm{ob}(\mathbf{C})$ and morphisms $f : B \to C$ and $g : C \to A$, then $f \circ 0 = 0 = 0 \circ g$. Note that each 0 here corresponds to a different zero morphism in different hom-spaces.

Proof Since $\mathbf{C}$ is $\mathbb{C}$-linear, composition is bilinear, hence in particular linear in each variable.

First, fix $f\colon B \to C$. Consider the map

$$\mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C), \qquad h \longmapsto f \circ h. \tag{1}$$

Linearity in the second variable gives

$$f \circ 0_{A,B} = f \circ (0_{A,B} + 0_{A,B}) = f \circ 0_{A,B} + f \circ 0_{A,B}. \tag{2}$$

Subtracting $f \circ 0_{A,B}$ from both sides in the abelian group underlying the vector space $\mathrm{Hom}(A, C)$ yields $f \circ 0_{A,B} = 0_{A,C}$.

Second, fix $g\colon C \to A$. Consider the map

$$\operatorname{Hom}(B, A) \longrightarrow \operatorname{Hom}(B, C), \qquad k \longmapsto k \circ g. \tag{3}$$

Linearity in the first variable gives

$$0_{B,A} \circ g = (0_{B,A} + 0_{B,A}) \circ g = 0_{B,A} \circ g + 0_{B,A} \circ g. \tag{4}$$

Subtracting $0_{B,A} \circ g$ from both sides in $\operatorname{Hom}(B, C)$ yields $0_{B,A} \circ g = 0_{B,C}$.

Thus composing with a zero morphism on either side yields the appropriate zero morphism in the target hom-space.

Observation Let C be a $\mathbb{C}-$linear category and let $A \in \operatorname{ob}(\mathrm{C})$. Then the following are equivalent

1. $\operatorname{Hom}(A, B) = 0, \forall B \in \operatorname{ob}(\mathrm{C})$
2. $\operatorname{Hom}(B, A) = 0, \forall B \in \operatorname{ob}(\mathrm{C})$
3. $\operatorname{Hom}(A, A) = 0$
4. $\mathbb{1}_A = 0$.

Proof We show the implications

$$(1) \Rightarrow (3), \qquad (2) \Rightarrow (3), \qquad (3) \Rightarrow (4), \qquad (4) \Rightarrow (1), \qquad (4) \Rightarrow (2).$$

(1) $\Rightarrow$ (3): taking $B = A$ in (1) gives $\operatorname{Hom}(A, A) = 0$.

(2) $\Rightarrow$ (3): taking $B = A$ in (2) gives $\operatorname{Hom}(A, A) = 0$.

(3) $\Rightarrow$ (4): since $\mathbb{1}_A \in \operatorname{Hom}(A, A)$ and this vector space is zero, we must have $\mathbb{1}_A = 0$.

(4) $\Rightarrow$ (1): let B be any object and $f \in \operatorname{Hom}(A, B)$. By the identity axiom,

$$f = f \circ \mathbb{1}_A, \tag{5}$$

and since $\mathbb{1}_A = 0$ and composition is bilinear, we obtain $f = f \circ 0 = 0$. Hence $\operatorname{Hom}(A, B) = 0$ for all B.

(4) $\Rightarrow$ (2): let B be any object and $g \in \operatorname{Hom}(B, A)$. Again by the identity axiom,

$$g = \mathbb{1}_A \circ g, \tag{6}$$

and since $\mathbb{1}_A = 0$, bilinearity of composition gives $g = 0 \circ g = 0$. Hence $\operatorname{Hom}(B, A) = 0$ for all B.

This proves the equivalence of all four conditions.

Definition (0−object) An object obeying any of the previous properties is called a 0−object.

Observation (There is a unique $0-$object up to unique isomorphism) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category. Suppose 0 and $0'$ are zero objects in the sense that for every object X one has

$$\mathrm{Hom}(0, X) = 0 = \mathrm{Hom}(X, 0) \qquad \text{and} \qquad \mathrm{Hom}(0', X) = 0 = \mathrm{Hom}(X, 0'). \quad (7)$$

Then there exists a unique isomorphism $0 \simeq 0'$.

Proof Since $\mathrm{Hom}(0, 0') = 0$, there is exactly one morphism $f : 0 \to 0'$, namely the zero morphism $f = 0_{0,0'}$. Likewise, since $\mathrm{Hom}(0', 0) = 0$, there is exactly one morphism $g : 0' \to 0$, namely $g = 0_{0',0}$.

Also, because $\mathrm{Hom}(0, 0) = 0$, we have $\mathbb{1}_0 \in \mathrm{Hom}(0, 0) = \{0_{0,0}\}$, hence

$$\mathbb{1}_0 = 0_{0,0}. \quad (8)$$

Similarly,

$$\mathbb{1}_{0'} = 0_{0',0'}. \quad (9)$$

Now $g \circ f \in \mathrm{Hom}(0, 0)$, but $\mathrm{Hom}(0, 0) = \{0_{0,0}\}$, so

$$g \circ f = 0_{0,0} = \mathbb{1}_0. \quad (10)$$

Likewise $f \circ g \in \mathrm{Hom}(0', 0') = \{0_{0',0'}\}$, hence

$$f \circ g = 0_{0',0'} = \mathbb{1}_{0'}. \quad (11)$$

Therefore f is an isomorphism with inverse g.

To prove uniqueness, note that any morphism $h : 0 \to 0'$ lies in $\mathrm{Hom}(0, 0') = 0$, hence $h = 0_{0,0'} = f$. So there is exactly one morphism $0 \to 0'$, and therefore exactly one isomorphism $0 \simeq 0'$.

Observation (The zero object really is the zero object) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category and let $0 \in \mathrm{ob}(\mathbf{C})$. Then the following are equivalent:

1. For every object X one has $\mathrm{Hom}(0, X) = 0$ and $\mathrm{Hom}(X, 0) = 0$ (zero vector spaces).

2. 0 is a zero object in the usual categorical sense (i.e. it is both initial and terminal)

Proof For any object X, the condition $\mathrm{Hom}(0, X) = 0$ means that the hom-space is the zero vector space, hence it has exactly one element. Therefore there exists a unique morphism $0 \to X$, so 0 is initial. Similarly, $\mathrm{Hom}(X, 0) = 0$ implies there is a unique morphism $X \to 0$, so 0 is terminal. Hence 0 is a zero object.

Assume 0 is a zero object, so for every object X the sets $\mathrm{Hom}(0, X)$ and $\mathrm{Hom}(X, 0)$ are singletons (unique morphisms by initiality and terminality). But in a $\mathbb{C}-$linear category each $\mathrm{Hom}(-, -)$ is a $\mathbb{C}-$vector space. A $\mathbb{C}-$vector space with exactly one element must be the zero vector space. Hence

$$\mathrm{Hom}(0, X) = 0 \quad \text{and} \quad \mathrm{Hom}(X, 0) = 0$$

as vector spaces for all X.

Definition (Direct sum of vector spaces) Let V and W be $\mathbb{C}-$vector spaces. The direct sum of V and W is the vector space

$$V \oplus W \ := \ \{(v,w) \mid v \in V,\ w \in W\},$$

with addition and scalar multiplication defined component-wise

$$(v,w) + (v',w') := (v+v',\ w+w'), \qquad \lambda(v,w) := (\lambda v,\ \lambda w). \tag{12}$$

.

Definition (Canonical embeddings and projections) Define linear maps

$$\iota_V : V \to V \oplus W, \quad \iota_V(v) := (v,0), \qquad \iota_W : W \to V \oplus W, \quad \iota_W(w) := (0,w),$$

$$\pi_V : V \oplus W \to V, \quad \pi_V(v,w) := v, \qquad \pi_W : V \oplus W \to W, \quad \pi_W(v,w) := w.$$

Lemma (Direct sum identities) The maps $\iota_V, \iota_W, \pi_V, \pi_W$ satisfy

$$\pi_V \circ \iota_V = \mathrm{id}_V, \qquad \pi_W \circ \iota_W = \mathrm{id}_W, \tag{13}$$

$$\pi_V \circ \iota_W = 0, \qquad \pi_W \circ \iota_V = 0, \tag{14}$$

$$\iota_V \circ \pi_V \ + \ \iota_W \circ \pi_W \ = \ \mathrm{id}_{V \oplus W}. \tag{15}$$

Proof For $v \in V$,

$$(\pi_V \circ \iota_V)(v) = \pi_V(v,0) = v, \tag{16}$$

so $\pi_V \circ \iota_V = \mathrm{id}_V$. For $w \in W$,

$$(\pi_W \circ \iota_W)(w) = \pi_W(0,w) = w, \tag{17}$$

so $\pi_W \circ \iota_W = \mathrm{id}_W$.

For $w \in W$,

$$(\pi_V \circ \iota_W)(w) = \pi_V(0,w) = 0, \tag{18}$$

hence $\pi_V \circ \iota_W = 0$. Similarly, for $v \in V$,

$$(\pi_W \circ \iota_V)(v) = \pi_W(v,0) = 0, \tag{19}$$

hence $\pi_W \circ \iota_V = 0$.

Finally, for $(v,w) \in V \oplus W$ we compute

$$\big((\iota_V \circ \pi_V) + (\iota_W \circ \pi_W)\big)(v,w) = \iota_V(\pi_V(v,w)) + \iota_W(\pi_W(v,w)) = \iota_V(v) + \iota_W(w) = (v,0) + (0,w) = (v,w). \tag{20}$$

Therefore $(\iota_V \circ \pi_V) + (\iota_W \circ \pi_W) = \mathrm{id}_{V \oplus W}$.

Definiton (Direct sum datum) Let V, W be $\mathbb{C}-$vector spaces. A direct sum datum for (V, W) consists of a vector space X and linear maps

$$\iota_V : V \to X, \qquad \iota_W : W \to X, \qquad \pi_V : X \to V, \qquad \pi_W : X \to W \tag{21}$$

such that

$$\pi_V \circ \iota_V = \mathrm{id}_V, \quad \pi_W \circ \iota_W = \mathrm{id}_W, \quad \pi_V \circ \iota_W = 0, \quad \pi_W \circ \iota_V = 0, \quad \iota_V \circ \pi_V + \iota_W \circ \pi_W = \mathrm{id}_X.$$

Proposition (Direct sum data characterize direct sums) If $(X, \iota_V, \iota_W, \pi_V, \pi_W)$ is a direct sum datum for (V, W), then there is a canonical isomorphism

$$\Phi : X \xrightarrow{\simeq} V \oplus W$$

given by $\Phi(x) = (\pi_V(x), \pi_W(x))$, with inverse

$$\Psi : V \oplus W \to X, \qquad \Psi(v, w) = \iota_V(v) + \iota_W(w).$$

Moreover, Φ and Ψ are mutually inverse and are compatible with the structure maps

$$\Phi \circ \iota_V(v) = (v, 0), \quad \Phi \circ \iota_W(w) = (0, w), \quad \pi_V = \mathrm{pr}_V \circ \Phi, \quad \pi_W = \mathrm{pr}_W \circ \Phi, \tag{22}$$

where $\mathrm{pr}_V, \mathrm{pr}_W$ are the standard projections from $V \oplus W$.

Proof Define linear maps

$$\Phi : X \to V \oplus W, \qquad \Phi(x) := (\pi_V(x), \pi_W(x)), \tag{23}$$

$$\Psi : V \oplus W \to X, \qquad \Psi(v, w) := \iota_V(v) + \iota_W(w). \tag{24}$$

Linearity of Φ is clear since π_V, π_W are linear. Linearity of Ψ follows from linearity of ι_V, ι_W and bilinearity of addition in X.

We show $\Phi \circ \Psi = \mathrm{id}_{V \oplus W}$. For $(v, w) \in V \oplus W$,

$$(\Phi \circ \Psi)(v, w) = \Phi(\iota_V(v) + \iota_W(w)) = \big(\pi_V(\iota_V(v) + \iota_W(w)),\ \pi_W(\iota_V(v) + \iota_W(w))\big). \tag{25}$$

Using linearity of π_V, π_W and the defining identities,

$$\pi_V(\iota_V(v) + \iota_W(w)) = \pi_V \iota_V(v) + \pi_V \iota_W(w) = v + 0 = v, \tag{26}$$

$$\pi_W(\iota_V(v) + \iota_W(w)) = \pi_W \iota_V(v) + \pi_W \iota_W(w) = 0 + w = w. \tag{27}$$

Hence $(\Phi \circ \Psi)(v, w) = (v, w)$.

Next we show $\Psi \circ \Phi = \mathrm{id}_X$. For $x \in X$,

$$(\Psi \circ \Phi)(x) = \Psi(\pi_V(x), \pi_W(x)) = \iota_V(\pi_V(x)) + \iota_W(\pi_W(x)) = \big((\iota_V \circ \pi_V) + (\iota_W \circ \pi_W)\big)(x) = \mathrm{id}_X(x) = x, \tag{28}$$

using the final defining identity of a direct sum datum. Therefore Φ and Ψ are inverse isomorphisms.

Compatibility with the structure maps follows by direct computation. For $v \in V$,

$$(\Phi \circ \iota_V)(v) = (\pi_V \iota_V(v), \pi_W \iota_V(v)) = (v, 0), \tag{29}$$

and similarly $(\Phi \circ \iota_W)(w) = (0, w)$. Finally, by definition of Φ we have $\pi_V = \mathrm{pr}_V \circ \Phi$ and $\pi_W = \mathrm{pr}_W \circ \Phi$.

Definition (Direct sum in a $\mathbb{C}-$linear category) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category and let $A_1, \ldots, A_n \in \mathrm{ob}(\mathbf{C})$. A direct sum of $A_1, \ldots, A_n$ is an object $A \in \mathrm{ob}(\mathbf{C})$ equipped with morphisms

$$\iota_i : A_i \to A, \qquad \pi_i : A \to A_i \qquad (1 \leq i \leq n) \tag{30}$$

such that the following $(n^2 + 1)$ equations hold:

$$\pi_i \circ \iota_j = \delta_{ij} \cdot \mathrm{id}_{A_j} \qquad (1 \leq i, j \leq n),$$

$$\sum_{j=1}^{n} \iota_j \circ \pi_j = \mathrm{id}_A.$$

Proposition Let $\mathbf{C}$ be a $\mathbb{C}-$linear category in which a zero object 0 exists. Then for every object $A \in \mathrm{ob}(\mathbf{C})$ one has

$$A \oplus 0 \;\simeq\; A \;\simeq\; 0 \oplus A,$$

in the sense that A is a direct sum of A and 0, and also a direct sum of 0 and A.

Proof Fix $A \in \mathrm{ob}(\mathbf{C})$ and let 0 be a zero object. By definition of zero object in a $\mathbb{C}-$linear category,

$$\mathrm{Hom}(0, A) = 0, \qquad \mathrm{Hom}(A, 0) = 0, \qquad \mathrm{End}(0) = \mathrm{Hom}(0, 0) = 0. \tag{31}$$

In particular, the unique morphisms $0 \to A$, $A \to 0$, and $0 \to 0$ are the corresponding zero morphisms, and

$$\mathrm{id}_0 = 0_{0,0}. \tag{32}$$

Claim 1: A is a direct sum of A and 0.

Define structure maps

$$\iota_1 := \mathrm{id}_A : A \to A, \qquad \iota_2 := 0_{0,A} : 0 \to A, \tag{33}$$

$$\pi_1 := \mathrm{id}_A : A \to A, \qquad \pi_2 := 0_{A,0} : A \to 0. \tag{34}$$

We verify the defining equations for $n = 2$ with $(A_1, A_2) = (A, 0)$ and A as the candidate sum.

For $i = j = 1$,

$$\pi_1 \circ \iota_1 = \mathrm{id}_A \circ \mathrm{id}_A = \mathrm{id}_A = \delta_{11} \cdot \mathrm{id}_A. \tag{35}$$

For $i = 1, j = 2$,

$$\pi_1 \circ \iota_2 = \mathrm{id}_A \circ 0_{0,A} = 0_{0,A} = \delta_{12} \cdot \mathrm{id}_0 = 0. \tag{36}$$

For $i = 2, j = 1$,

$$\pi_2 \circ \iota_1 = 0_{A,0} \circ \mathrm{id}_A = 0_{A,0} = \delta_{21} \cdot \mathrm{id}_A = 0. \tag{37}$$

For $i = j = 2$,

$$\pi_2 \circ \iota_2 = 0_{A,0} \circ 0_{0,A} = 0_{0,0} = \mathrm{id}_0 = \delta_{22} \cdot \mathrm{id}_0, \tag{38}$$

using that composition with a zero morphism yields a zero morphism and that $\mathrm{id}_0 = 0_{0,0}$.

Finally,

$$\iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_A \circ \mathrm{id}_A + 0_{0,A} \circ 0_{A,0} = \mathrm{id}_A + 0_{A,A} = \mathrm{id}_A. \tag{39}$$

Thus A is a direct sum of A and 0, hence $A \oplus 0 \simeq A$.

Claim 2: A is a direct sum of 0 and A.

Now take $(A_1, A_2) = (0, A)$ and again use A as the candidate sum. Define

$$\iota_1 := 0_{0,A} : 0 \to A, \qquad \iota_2 := \mathrm{id}_A : A \to A, \tag{40}$$

$$\pi_1 := 0_{A,0} : A \to 0, \qquad \pi_2 := \mathrm{id}_A : A \to A. \tag{41}$$

The same computations as above (with indices exchanged) show that

$$\pi_i \circ \iota_j = \delta_{ij} \cdot \mathrm{id}_{A_j} \quad (1 \le i, j \le 2), \qquad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_A. \tag{42}$$

Hence A is a direct sum of 0 and A, i.e. $0 \oplus A \simeq A$.

Combining the two claims yields

$$A \oplus 0 \simeq A \simeq 0 \oplus A. \tag{43}$$

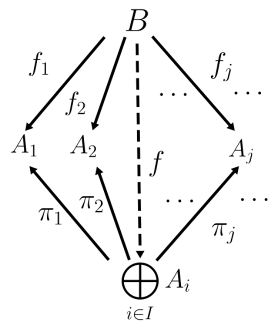
Proposition (Direct sums satisfy the universal property of products) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category and let $A_1, \ldots, A_n \in \mathrm{ob}(\mathbf{C})$. Suppose $(A, \{\iota_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n)$ is a direct sum of $A_1, \ldots, A_n$, i.e.

$$\pi_i \circ \iota_j = \delta_{ij} \, \mathrm{id}_{A_j} \quad (1 \le i, j \le n), \qquad \sum_{j=1}^{n} \iota_j \circ \pi_j = \mathrm{id}_A.$$

Then A satisfies the universal property of a product: for every object $B \in \mathbf{C}$ and morphisms $f_i : B \to A_i$ $(1 \le i \le n)$, there exists a unique morphism $f : B \to A$ such that

$$\pi_i \circ f = f_i \qquad (1 \le i \le n). \tag{44}$$

Diagramatically this amounts to showing that the following diagram commutes



Proof

Existence Define

$$f \;\equiv\; \sum_{j=1}^{n} \iota_j \circ f_j \;\in\; \mathrm{Hom}(B, A). \tag{45}$$

For each i, using bilinearity of composition,

$$\pi_i \circ f = \pi_i \circ \Big(\sum_{j=1}^{n} \iota_j \circ f_j\Big) = \sum_{j=1}^{n} (\pi_i \circ \iota_j) \circ f_j = \sum_{j=1}^{n} \delta_{ij}\, \mathrm{id}_{A_j} \circ f_j = f_i. \tag{46}$$

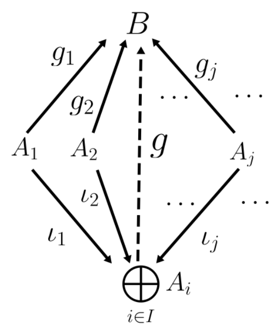
Uniqueness If $f' : B \to A$ also satisfies $\pi_i \circ f' = f_i$ for all i, then

$$f' = \mathrm{id}_A \circ f' = \Big(\sum_{j=1}^{n} \iota_j \circ \pi_j\Big) \circ f' = \sum_{j=1}^{n} \iota_j \circ (\pi_j \circ f') = \sum_{j=1}^{n} \iota_j \circ f_j = f. \tag{47}$$

Proposition (Direct sums satisfy the universal property of coproducts) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category and let $A_1, \ldots, A_n \in \mathrm{ob}(\mathbf{C})$. Suppose $(A, \{\iota_i\}_{i=1}^n, \{\pi_i\}_{i=1}^n)$ is a direct sum of $A_1, \ldots, A_n$. Then A satisfies the universal property of a coproduct: for every object $C \in \mathbf{C}$ and morphisms $g_i : A_i \to C$ $(1 \le i \le n)$, there exists a unique morphism $g : A \to C$ such that

$$g \circ \iota_i = g_i \qquad (1 \le i \le n). \tag{48}$$

Diagramatically this amounts to showing that the following diagram commutes



Proof

Existence Define

$$g \ := \ \sum_{j=1}^{n} g_j \circ \pi_j \ \in \ \mathrm{Hom}(A, C). \tag{49}$$

Then for each i,

$$g \circ \iota_i = \Big(\sum_{j=1}^{n} g_j \circ \pi_j\Big) \circ \iota_i = \sum_{j=1}^{n} g_j \circ (\pi_j \circ \iota_i) = \sum_{j=1}^{n} g_j \circ \big(\delta_{ji}\,\mathrm{id}_{A_i}\big) = g_i. \tag{50}$$

Uniqueness If $g' : A \to C$ also satisfies $g' \circ \iota_i = g_i$ for all i, then

$$g' = g' \circ \mathrm{id}_A = g' \circ \Big(\sum_{j=1}^{n} \iota_j \circ \pi_j\Big) = \sum_{j=1}^{n}(g' \circ \iota_j) \circ \pi_j = \sum_{j=1}^{n} g_j \circ \pi_j = g. \tag{51}$$

Theorem (Representation of $\mathbb{C}-$linear morphisms as matrices) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category. Let

$$A := A_1 \oplus \cdots \oplus A_n, \qquad B := B_1 \oplus \cdots \oplus B_m \tag{52}$$

be direct sums with structure maps

$$\iota_i : A_i \to A,\ \pi_i : A \to A_i\ (1 \le i \le n), \qquad \jmath_j : B_j \to B,\ \rho_j : B \to B_j\ (1 \le j \le m). \tag{53}$$

Then giving a morphism $f : A \to B$ is equivalent to giving an $m \times n$ matrix of morphisms

$$(f_{ji}) \quad \text{with} \quad f_{ji} : A_i \to B_j, \tag{54}$$

via the relations

$$f_{ji} \ = \ \rho_j \circ f \circ \iota_i, \tag{55}$$

and conversely,

$$f \ = \ \sum_{j=1}^{m}\sum_{i=1}^{n} \jmath_j \circ f_{ji} \circ \pi_i. \tag{56}$$

Proof Given $f : A \to B$, define $f_{ji} := \rho_j \circ f \circ \iota_i : A_i \to B_j$.

Conversely, given morphisms $f_{ji} : A_i \to B_j$, define

$$f \; := \; \sum_{j=1}^{m} \sum_{i=1}^{n} \jmath_j \circ f_{ji} \circ \pi_i \; \in \; \mathrm{Hom}(A, B). \tag{57}$$

Using bilinearity and the direct sum identities $\rho_j \circ \jmath_k = \delta_{jk} \, \mathrm{id}_{B_k}$ and $\pi_\ell \circ \iota_i = \delta_{\ell i} \, \mathrm{id}_{A_i}$, we compute

$$\rho_j \circ f \circ \iota_i = \rho_j \circ \Big(\sum_{k=1}^{m} \sum_{\ell=1}^{n} \jmath_k \circ f_{k\ell} \circ \pi_\ell \Big) \circ \iota_i = \sum_{k,\ell} (\rho_j \circ \jmath_k) \circ f_{k\ell} \circ (\pi_\ell \circ \iota_i) = \sum_{k,\ell} \delta_{jk} \delta_{\ell i} \, f_{k\ell} = f_{ji}. \tag{58}$$

Finally, starting from a given f, forming $f_{ji} = \rho_j f \iota_i$ and reconstructing f by the above formula yields the original morphism, by the product universal property of B. Hence $\mathrm{Hom}(A, B)$ is canonically identified with the set of $m \times n$ matrices (f_{ji}) with entries $f_{ji} \in \mathrm{Hom}(A_i, B_j)$.

Definition (Direct sum of ℂ−linear categories) Let $\mathbf{C}$ and $\mathbf{D}$ be ℂ−linear categories. The direct sum of $\mathbf{C}$ and $\mathbf{D}$, denoted by $\mathbf{C} \oplus \mathbf{D}$, is the ℂ−linear category defined as follows.

1. Objects: The objects of $\mathbf{C} \oplus \mathbf{D}$ are pairs
$$\mathrm{ob}(\mathbf{C} \oplus \mathbf{D}) := \mathrm{ob}(\mathbf{C}) \times \mathrm{ob}(\mathbf{D}). \tag{59}$$
2. Morphisms: For objects $(A, B), (A', B') \in \mathrm{ob}(\mathbf{C} \oplus \mathbf{D})$, the hom-space is
$$\mathrm{Hom}_{\mathbf{C} \oplus \mathbf{D}}\big((A, B), (A', B')\big) \equiv \mathrm{Hom}_{\mathbf{C}}(A, A') \oplus \mathrm{Hom}_{\mathbf{D}}(B, B'), \tag{60}$$
where the right-hand side is the direct sum of ℂ−vector spaces.
3. Composition: Given morphisms
$$(f, g) \in \mathrm{Hom}_{\mathbf{C} \oplus \mathbf{D}}\big((A, B), (A', B')\big), \qquad (f', g') \in \mathrm{Hom}_{\mathbf{C} \oplus \mathbf{D}}\big((A', B'), (A'', B'')\big), \tag{61}$$
their composite is defined componentwise by
$$(f', g') \circ (f, g) := (f' \circ f, \; g' \circ g), \tag{62}$$
where $f' \circ f$ is composition in $\mathbf{C}$ and $g' \circ g$ is composition in $\mathbf{D}$.
4. Identity morphisms: For each object $(A, B) \in \mathrm{ob}(\mathbf{C} \oplus \mathbf{D})$, the identity morphism is
$$\mathrm{id}_{(A,B)} := (\mathrm{id}_A, \mathrm{id}_B). \tag{63}$$

With these definitions, $\mathbf{C} \oplus \mathbf{D}$ is a ℂ−linear category: each hom-space is a ℂ−vector space, and composition is bilinear since it is defined componentwise using the bilinear compositions in $\mathbf{C}$ and $\mathbf{D}$.

Proposition Let $\mathbf{C}, \mathbf{D}$ be $\mathbb{C}-$linear categories. Suppose $A \in \mathbf{C}$ is a direct sum of $A_1, \ldots, A_n \in \mathbf{C}$ with structure maps

$$\iota_i : A_i \to A, \qquad \pi_i : A \to A_i \qquad (1 \le i \le n), \tag{64}$$

and $B \in \mathbf{D}$ is a direct sum of $B_1, \ldots, B_n \in \mathbf{D}$ with structure maps

$$\kappa_i : B_i \to B, \qquad \rho_i : B \to B_i \qquad (1 \le i \le n). \tag{65}$$

Then $(A, B) \in \mathbf{C} \oplus \mathbf{D}$ is a direct sum of

$$(A_1, B_1), \ldots, (A_n, B_n) \in \mathbf{C} \oplus \mathbf{D} \tag{66}$$

with embeddings and projections given by

$$\widetilde{\iota}_i := (\iota_i, \kappa_i) : (A_i, B_i) \to (A, B), \qquad \widetilde{\pi}_i := (\pi_i, \rho_i) : (A, B) \to (A_i, B_i).$$

Proof Recall that in $\mathbf{C} \oplus \mathbf{D}$ we have

$$\mathrm{Hom}_{\mathbf{C}\oplus\mathbf{D}}\big((X, Y), (X', Y')\big) = \mathrm{Hom}_{\mathbf{C}}(X, X') \oplus \mathrm{Hom}_{\mathbf{D}}(Y, Y'), \tag{67}$$

and composition is componentwise

$$(f', g') \circ (f, g) = (f' \circ f,\ g' \circ g).$$

We must check the defining equations for a direct sum in $\mathbf{C} \oplus \mathbf{D}$.

(1) The $\delta-$relations For $1 \le i, j \le n$,

$$\widetilde{\pi}_i \circ \widetilde{\iota}_j = (\pi_i, \rho_i) \circ (\iota_j, \kappa_j) = (\pi_i \circ \iota_j,\ \rho_i \circ \kappa_j). \tag{68}$$

Since A is a direct sum of the A_j in $\mathbf{C}$ and B is a direct sum of the B_j in $\mathbf{D}$,

$$\pi_i \circ \iota_j = \delta_{ij}\, \mathrm{id}_{A_j}, \qquad \rho_i \circ \kappa_j = \delta_{ij}\, \mathrm{id}_{B_j}. \tag{69}$$

Hence

$$\widetilde{\pi}_i \circ \widetilde{\iota}_j = (\delta_{ij}\, \mathrm{id}_{A_j},\ \delta_{ij}\, \mathrm{id}_{B_j}) = \delta_{ij}\, (\mathrm{id}_{A_j}, \mathrm{id}_{B_j}) = \delta_{ij}\, \mathrm{id}_{(A_j, B_j)}. \tag{70}$$

(2) The splitting identity Using addition in the hom-space of $\mathbf{C} \oplus \mathbf{D}$ componentwise, we have

$$\sum_{j=1}^{n} \widetilde{\iota}_j \circ \widetilde{\pi}_j = \sum_{j=1}^{n} (\iota_j, \kappa_j) \circ (\pi_j, \rho_j) = \sum_{j=1}^{n} (\iota_j \circ \pi_j,\ \kappa_j \circ \rho_j) = \Big(\sum_{j=1}^{n} \iota_j \circ \pi_j,\ \sum_{j=1}^{n} \kappa_j \circ \rho_j\Big). \tag{71}$$

By the direct sum identities in $\mathbf{C}$ and $\mathbf{D}$,

$$\sum_{j=1}^{n} \iota_j \circ \pi_j = \mathrm{id}_A, \qquad \sum_{j=1}^{n} \kappa_j \circ \rho_j = \mathrm{id}_B. \tag{72}$$

Therefore

$$\sum_{j=1}^{n} \widetilde{\iota}_j \circ \widetilde{\pi}_j = (\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{(A,B)}. \tag{73}$$

Thus the morphisms $\widetilde{\iota}_i$ and $\widetilde{\pi}_i$ satisfy the defining equations for a direct sum of $(A_1, B_1), \ldots, (A_n, B_n)$ in $\mathbf{C} \oplus \mathbf{D}$. Hence (A, B) is their direct sum.

2 Simplicity and semisimplicity

Definition (Simple object) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category. An object $A \in \mathrm{ob}(\mathbf{C})$ is called simple if

$$\mathrm{Hom}_{\mathbf{C}}(A, A) \simeq \mathbb{C} \tag{74}$$

as $\mathbb{C}-$vector spaces.

Definition (Disjoint objects) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category. Two objects $A, B \in \mathrm{ob}(\mathbf{C})$ are called disjoint if

$$\mathrm{Hom}_{\mathbf{C}}(A, B) = 0 \qquad \text{and} \qquad \mathrm{Hom}_{\mathbf{C}}(B, A) = 0. \tag{75}$$

Definition (Semisimple and finite semisimple $\mathbb{C}-$linear categories) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category. We say that $\mathbf{C}$ is semisimple if it satisfies:

1. Finite direct sums exist in $\mathbf{C}$ (i.e. the direct sum of any finite family of objects exists).
2. There exists a collection of objects $\{A_i\}_{i \in I} \subset \mathrm{ob}(\mathbf{C})$ such that
   (a) each A_i is simple;
   (b) the A_i are mutually disjoint: for $i \neq j$,
   $$\mathrm{Hom}_{\mathbf{C}}(A_i, A_j) = 0 = \mathrm{Hom}_{\mathbf{C}}(A_j, A_i); \tag{76}$$
   (c) every object of $\mathbf{C}$ is isomorphic to a finite direct sum of objects among $\{A_i\}_{i \in I}$.

   If the index set I is finite, then $\mathbf{C}$ is called finite semisimple.

Definition (Indecomposable object) Let $\mathbf{C}$ be a $\mathbb{C}-$linear category. A nonzero object $A \in \mathrm{ob}(\mathbf{C})$ is called indecomposable if for every decomposition

$$A \simeq A_1 \oplus A_2 \tag{77}$$

one has $A_1 = 0$ or $A_2 = 0$

Observation Every simple object is indecomposable. In general, the converse need not hold.

Proposition Let $\mathbf{C}$ be a semisimple $\mathbb{C}-$linear category. Then an object $A \in \mathrm{ob}(\mathbf{C})$ is indecomposable if and only if it is simple.

Proof

(⇒) Simple $\implies$ indecomposable Suppose A is simple and admits a decomposition

$$A \simeq A_1 \oplus A_2. \tag{78}$$

Let $\iota_i : A_i \to A$ and $\pi_i : A \to A_i$ be the associated structure maps. Since A is simple, $\mathrm{End}_{\mathbf{C}}(A) \simeq \mathbb{C}$. In particular, every idempotent endomorphism of A is either 0 or id_A.

Consider the idempotents

$$e_1 := \iota_1 \circ \pi_1, \qquad e_2 := \iota_2 \circ \pi_2 \tag{79}$$

in $\mathrm{End}_{\mathbf{C}}(A)$. They satisfy

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 + e_2 = \mathrm{id}_A, \quad e_1 e_2 = e_2 e_1 = 0. \tag{80}$$

Since$\mathrm{End}_{\mathbf{C}}(A) \simeq \mathbb{C}$, one of e_1, e_2 must be zero. If $e_1 = 0$, then $\pi_1 = 0$ and hence $A_1 = 0$. Similarly, if $e_2 = 0$, then $A_2 = 0$. Thus the decomposition is trivial, and A is indecomposable.

(⇐) Indecomposable $\implies$ simple Assume now that $\mathbf{C}$ is semisimple and that A is indecomposable. By semisimplicity, there exists a collection of mutually disjoint simple objects $\{S_i\}_{i \in I}$ such that

$$A \simeq \bigoplus_{k=1}^{n} S_{i_k} \tag{81}$$

for some finite list $i_1, \ldots, i_n \in I$.

If $n \geq 2$, then A admits a nontrivial decomposition

$$A \simeq S_{i_1} \oplus \Big(\bigoplus_{k=2}^{n} S_{i_k} \Big), \tag{82}$$

with both summands nonzero, contradicting the indecomposability of A. Hence $n = 1$, and therefore

$$A \simeq S_{i_1}, \tag{83}$$

which is simple.

Thus, in a semisimple $\mathbb{C}-$linear category, an object is indecomposable if and only if it is simple.

Proposition (Block-diagonal form of morphisms in a semisimple category) Let $\mathbf{C}$ be a semisimple $\mathbb{C}-$linear category, and let $\{X_i\}_{i \in I}$ be a collection of mutually disjoint simple objects such that every object of $\mathbf{C}$ is a finite direct sum of objects in $\{X_i\}_{i \in I}$. For finite families of nonnegative integers $\{n_i\}_{i \in I}$ and $\{m_i\}_{i \in I}$, there is a canonical isomorphism of vector spaces

$$\mathrm{Hom}_{\mathbf{C}} \left(\bigoplus_{i \in I} X_i^{\oplus n_i}, \bigoplus_{j \in I} X_j^{\oplus m_j} \right) \cong \bigoplus_{i \in I} M_{m_i \times n_i}(\mathbb{C}), \tag{84}$$

where $M_{m \times n}(\mathbb{C})$ denotes the space of $m \times n$ matrices with entries in $\mathbb{C}$.

Proof Representation of $\mathbb{C}-$linear morphisms as matrices, a morphism

$$f : \bigoplus_{i\in I} X_i^{\oplus n_i} \longrightarrow \bigoplus_{j\in I} X_j^{\oplus m_j}$$

is equivalent to a matrix of morphisms

$$(f_{ji}) : X_i^{\oplus n_i} \to X_j^{\oplus m_j}. \tag{85}$$

Since each X_i is simple and the X_i are mutually disjoint, we have

$$\mathrm{Hom}_{\mathbf{C}}(X_i, X_j) = \begin{cases} \mathbb{C} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{86}$$

Therefore all off-diagonal blocks f_{ji} with $i \neq j$ vanish, and the only nonzero components are the diagonal ones

$$f_{ii} \in \mathrm{Hom}_{\mathbf{C}}(X_i^{\oplus n_i}, X_i^{\oplus m_i}). \tag{87}$$

Again the theorem of representation of $\mathbb{C}-$linear morphisms as matrices, each space $\mathrm{Hom}_{\mathbf{C}}(X_i^{\oplus n_i}, X_i^{\oplus m_i})$ is canonically isomorphic to $M_{m_i\times n_i}(\mathbb{C})$. Taking the direct sum over all $i \in I$ yields the claimed decomposition.

Proposition (Classification of simple objects) Let $\mathbf{C}$ be a semisimple $\mathbb{C}-$linear category and $\{X_i\}_{i\in I}$ a collection of mutually disjoint simple objects such that every object of $\mathbf{C}$ is a finite direct sum of objects in $\{X_i\}_{i\in I}$. Then every simple object of $\mathbf{C}$ is isomorphic to exactly one object in the family $\{X_i\}_{i\in I}$.

Proof Let S be a simple object in $\mathbf{C}$. By assumption, there exists a finite decomposition

$$S \simeq \bigoplus_{k=1}^{n} X_{i_k}. \tag{88}$$

Since S is simple and hence indecomposable, we must have $n = 1$. Thus $S \simeq X_i$ for some $i \in I$.

Uniqueness follows from disjointness: if $X_i \cong X_j$, then

$$\mathrm{Hom}_{\mathbf{C}}(X_i, X_j) \simeq \mathbb{C} \neq 0, \tag{89}$$

which forces $i = j$.

Proposition (Finiteness criterion for semisimplicity) Let $\mathbf{C}$ be a semisimple $\mathbb{C}-$linear category. Then $\mathbf{C}$ is finite semisimple if and only if there are finitely many isomorphism classes of simple objects in $\mathbf{C}$.

Proof If $\mathbf{C}$ is finite semisimple, by definition there exists a finite family $\{X_i\}_{i\in I}$ of mutually disjoint simple objects such that every object of $\mathbf{C}$ is a finite direct sum of these objects. Hence there are finitely many isomorphism classes of simple objects.

Conversely, suppose $\mathbf{C}$ has only finitely many isomorphism classes of simple objects. Choose one representative from each class to form a finite family $\{X_i\}_{i\in I}$. By semisimplicity, every object of $\mathbf{C}$ is a finite direct sum of simple objects, hence a finite direct sum of objects in $\{X_i\}_{i\in I}$. Therefore $\mathbf{C}$ is finite semisimple.

Definition (Simple spectrum) Let $\mathbf{C}$ be a semisimple $\mathbb{C}-$linear category. The set of isomorphism classes of simple objects in $\mathbf{C}$ is denoted by $\mathrm{Irr}(\mathbf{C})$.

Proposition Let $\mathbf{C}$ and $\mathbf{D}$ be semisimple $\mathbb{C}-$linear categories, and let $\mathbf{C}\oplus\mathbf{D}$ be their direct sum $\mathbb{C}-$linear category. Then:

1. $\mathbf{C}\oplus\mathbf{D}$ is semisimple.
2. The simple objects of $\mathbf{C}\oplus\mathbf{D}$ are exactly the objects of the form
$$(A,0) \text{ with } A \text{ simple in } \mathbf{C} \qquad \text{or} \qquad (0,B) \text{ with } B \text{ simple in } \mathbf{D}. \tag{90}$$
3. $\mathbf{C}\oplus\mathbf{D}$ is finite semisimple if and only if both $\mathbf{C}$ and $\mathbf{D}$ are finite semisimple.

Proof Recall that $\mathrm{ob}(\mathbf{C}\oplus\mathbf{D}) = \mathrm{ob}(\mathbf{C})\times\mathrm{ob}(\mathbf{D})$ and
$$\mathrm{Hom}_{\mathbf{C}\oplus\mathbf{D}}\big((A,B),(A',B')\big) = \mathrm{Hom}_{\mathbf{C}}(A,A')\oplus\mathrm{Hom}_{\mathbf{D}}(B,B'), \tag{91}$$
with componentwise composition.

(1) $\mathbf{C}\oplus\mathbf{D}$ is semisimple Since $\mathbf{C}$ and $\mathbf{D}$ are semisimple, finite direct sums exist in each. Given finitely many objects $(A_1,B_1),\dots,(A_n,B_n)$ in $\mathbf{C}\oplus\mathbf{D}$, let
$$A := A_1\oplus\cdots\oplus A_n \text{ in } \mathbf{C}, \qquad B := B_1\oplus\cdots\oplus B_n \text{ in } \mathbf{D}. \tag{92}$$
By the previously proved result about direct sums in $\mathbf{C}\oplus\mathbf{D}$,
$$(A,B) \text{ is a direct sum of } (A_1,B_1),\dots,(A_n,B_n). \tag{93}$$
Hence finite direct sums exist in $\mathbf{C}\oplus\mathbf{D}$.

Next, since $\mathbf{C}$ is semisimple, choose a collection of mutually disjoint simple objects$\{X_i\}_{i\in I}\subset\mathrm{ob}(\mathbf{C})$ such that every object of $\mathbf{C}$ is a finite direct sum of the X_i. Similarly choose $\{Y_j\}_{j\in J}\subset\mathrm{ob}(\mathbf{D})$ with the analogous property in $\mathbf{D}$.

Consider the collection in $\mathbf{C}\oplus\mathbf{D}$:
$$\{(X_i,0)\}_{i\in I} \,\cup\, \{(0,Y_j)\}_{j\in J}. \tag{94}$$

These are mutually disjoint: if $i \neq i'$ then

$$\mathrm{Hom}_{\mathbf{C}\oplus\mathbf{D}}((X_i, 0), (X_{i'}, 0)) = \mathrm{Hom}_{\mathbf{C}}(X_i, X_{i'}) \oplus \mathrm{Hom}_{\mathbf{D}}(0, 0) = 0, \tag{95}$$

and similarly in the other cases, using disjointness in $\mathbf{C}$ and $\mathbf{D}$ and that $\mathrm{Hom}(0, -) = 0 = \mathrm{Hom}(-, 0)$.

Every object (A, B) of $\mathbf{C} \oplus \mathbf{D}$ decomposes as

$$(A, B) \ \simeq \ (A, 0) \oplus (0, B),$$

and then A and B decompose as finite direct sums of X_i and Y_j, so (A, B) is a finite direct sum of objects among $(X_i, 0)$ and $(0, Y_j)$. Thus $\mathbf{C}\oplus\mathbf{D}$ is semisimple.

(2) Classification of simple objects in $\mathbf{C} \oplus \mathbf{D}$ First, if A is simple in $\mathbf{C}$, then

$$\mathrm{End}_{\mathbf{C}\oplus\mathbf{D}}((A, 0)) = \mathrm{End}_{\mathbf{C}}(A) \oplus \mathrm{End}_{\mathbf{D}}(0) \simeq \mathbb{C} \oplus 0 \simeq \mathbb{C},$$

so $(A, 0)$ is simple in $\mathbf{C}\oplus\mathbf{D}$. Similarly, if B is simple in $\mathbf{D}$, then $(0, B)$ is simple.

Conversely, let (A, B) be a simple object in $\mathbf{C}\oplus\mathbf{D}$. Using the decomposition

$$(A, B) \ \simeq \ (A, 0) \oplus (0, B), \tag{96}$$

simplicity implies indecomposability, hence one of the summands must be zero: either $(A, 0) = 0$ or $(0, B) = 0$. Thus either $A = 0$ or $B = 0$.

If $(A, B) \simeq (A, 0)$ with $A \neq 0$, then

$$\mathrm{End}_{\mathbf{C}\oplus\mathbf{D}}((A, 0)) \simeq \mathrm{End}_{\mathbf{C}}(A),$$

so $\mathrm{End}_{\mathbf{C}}(A) \simeq \mathbb{C}$, i.e. A is simple in $\mathbf{C}$. Similarly, if $(A, B) \simeq (0, B)$ with $B \neq 0$, then B is simple in $\mathbf{D}$. Therefore the simple objects are precisely those of the stated two forms.

(3) Finiteness By (2), isomorphism classes of simple objects in $\mathbf{C} \oplus \mathbf{D}$ are exactly the disjoint union of isomorphism classes of simples in $\mathbf{C}$ and in $\mathbf{D}$:

$$\mathrm{Irr}(\mathbf{C} \oplus \mathbf{D}) \ \simeq \ \mathrm{Irr}(\mathbf{C}) \sqcup \mathrm{Irr}(\mathbf{D}).$$

Hence $\mathbf{C} \oplus \mathbf{D}$ has finitely many isomorphism classes of simples if and only if both $\mathbf{C}$ and $\mathbf{D}$ do. By the finiteness criterion for semisimple categories, this is equivalent to $\mathbf{C} \oplus \mathbf{D}$ being finite semisimple if and only if both $\mathbf{C}$ and $\mathbf{D}$ are finite semisimple.

Definition (Idempotent) Let $\mathbf{C}$ be a category and let $A \in \mathrm{ob}(\mathbf{C})$. A morphism

$$e : A \to A \tag{97}$$

is called an idempotent if

$$e \circ e = e. \tag{98}$$

Definition (Idempotent complete category) A category $\mathbf{C}$ is called idempotent complete (or Karoubian) if for every object $A \in \mathrm{ob}(\mathbf{C})$ and every idempotent $e : A \to A$, there exist an object $B \in \mathrm{ob}(\mathbf{C})$ and morphisms

$$i : B \to A, \qquad p : A \to B \tag{99}$$

such that

$$p \circ i = \mathrm{id}_B, \qquad i \circ p = e. \tag{100}$$

Equivalently, every idempotent endomorphism in $\mathbf{C}$ arises as the projection onto a direct summand.

Theorem (Equivalence of categories) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor between categories $\mathbf{C}$ and $\mathbf{D}$. Then F is an equivalence of categories if and only if the following conditions hold:

1. (Fully faithful) For any objects $A, B \in \mathbf{C}$, the map

   $$F_{A,B} : \mathrm{Hom}_{\mathbf{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{D}}(F(A), F(B)), \qquad f \longmapsto F(f), \tag{101}$$

   is a bijection.

2. (Essentially surjective) For any object $D \in \mathbf{D}$, there exists an object $A \in \mathbf{C}$ such that $F(A) \simeq D$.

Observation In many contexts (e.g. $\mathbb{C}-$linear categories), "bijection" in (1) can be strengthened to "isomorphism of $\mathbb{C}-$vector spaces" when the functor is $\mathbb{C}-$linear.

Definition ($\mathrm{Vec}^{\oplus n}$)

Let Vec denote the $\mathbb{C}-$linear category of finite-dimensional $\mathbb{C}-$vector spaces. For $n \geq 0$, define

$$\mathrm{Vec}^{\oplus n} := \underbrace{\mathrm{Vec} \oplus \cdots \oplus \mathrm{Vec}}_{n \text{ copies}}, \tag{102}$$

the direct sum of $\mathbb{C}-$linear categories (so objects are n-tuples of vector spaces, and hom-spaces are direct sums of hom-spaces componentwise).

Proposition (Finite semisimple categories are equivalent to $\mathrm{Vec}^{\oplus n}$) Let $\mathbf{C}$ be a finite semisimple $\mathbb{C}-$linear category, and let

$$n := \left|\mathrm{Irr}(\mathbf{C})\right| \tag{103}$$

be the number of isomorphism classes of simple objects in $\mathbf{C}$. Then $\mathbf{C}$ is equivalent (as a $\mathbb{C}-$linear category) to $\mathrm{Vec}^{\oplus n}$.

Proof Choose representatives $S_1, \ldots, S_n$ for the isomorphism classes of simple objects in $\mathbf{C}$. Define a $\mathbb{C}-$linear functor

$$F : \mathrm{Vec}^{\oplus n} \longrightarrow \mathbf{C} \tag{104}$$

on objects by

$$F(V_1, \ldots, V_n) \equiv \bigoplus_{i=1}^{n} \big(S_i \otimes_{\mathbb{C}} V_i\big), \tag{105}$$

where $S_i \otimes_{\mathbb{C}} V_i$ denotes the V_i-fold direct sum of S_i (made precise by choosing a basis of V_i, or equivalently by defining $S_i \otimes V_i \equiv S_i^{\oplus \dim V_i}$ functorially). On morphisms, F acts componentwise: a tuple of linear maps $\alpha_i : V_i \to W_i$ is sent to the induced morphism $S_i \otimes V_i \to S_i \otimes W_i$ and then summed over i.

Because the S_i are mutually disjoint simples, morphisms between different S_i-summands vanish, and the endomorphisms of each S_i are canonically $\mathbb{C}$. Hence morphisms between

$$\bigoplus_i (S_i \otimes V_i) \quad \text{and} \quad \bigoplus_i (S_i \otimes W_i) \tag{106}$$

identify with $\bigoplus_i \mathrm{Hom}_{\mathbb{C}}(V_i, W_i)$, which is exactly the hom-space in $\mathrm{Vec}^{\oplus n}$. This shows F is fully faithful.

Essential surjectivity follows from semisimplicity: every object of $\mathbf{C}$ is a finite direct sum $\bigoplus_i S_i^{\oplus m_i}$, which is isomorphic to $F(\mathbb{C}^{m_1}, \ldots, \mathbb{C}^{m_n})$.

Therefore F is fully faithful and essentially surjective, hence an equivalence by the equivalence of categories theorem.

\begin{corollary}[Finite semisimple categories are idempotent complete] Every finite semisimple $\mathbb C$-linear category is idempotent complete. \end{corollary}

Proof

Since $\mathbf{C} \simeq \mathrm{Vec}^{\oplus n}$ by Proposition~\ref{prop:finss-vecn}, it suffices to note that Vec is idempotent complete: if $e : V \to V$ is an idempotent linear map, then

$$V \simeq \mathrm{im}(e) \oplus \ker(e)$$

and e is the projection onto $\mathrm{im}(e)$. Finite direct sums preserve idempotent completeness, so $\mathrm{Vec}^{\oplus n}$ is idempotent complete, and equivalences preserve this property.

Observation (Alternative definition of finite semisimplicity) In practice, one may define a finite semisimple $\mathbb{C}-$linear category to be a $\mathbb{C}-$linear category equivalent to $\mathrm{Vec}^{\oplus n}$ for some $n \geq 0$.

3 Interaction of $\mathbb{C}-$linearity and semisimplicity with other structure

Definition ($\mathbb{C}-$linear monoidal category) A $\mathbb{C}-$linear monoidal category $\mathbf{C}$ is a category equipped with the following data:

1. a $\mathbb{C}-$linear category structure on $\mathbf{C}$, i.e.\ for all objects $A, B \in \mathrm{ob}(\mathbf{C})$ the hom-space $\mathrm{Hom}_{\mathbf{C}}(A, B)$ is a $\mathbb{C}-$vector space and composition is $\mathbb{C}-$bilinear;
2. a monoidal structure $(\mathbf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, where
$$\otimes : \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C} \tag{107}$$
is a bifunctor, $\mathbb{1}$ is the unit object, and α, λ, ρ are the usual associativity and unit isomorphisms;
3. the tensor product is $\mathbb{C}-$bilinear: for each object $A \in \mathrm{ob}(\mathbf{C})$, the functors
$$A \otimes - : \mathbf{C} \to \mathbf{C} \qquad \text{and} \qquad - \otimes A : \mathbf{C} \to \mathbf{C} \tag{108}$$
are $\mathbb{C}-$linear functors.

Definition (Spatial fusion anomaly) Let $\mathbf{C}$ be a $\mathbb{C}-$linear monoidal category. For any objects $A, A', B, B' \in \mathrm{ob}(\mathbf{C})$, the tensor product bifunctor induces a map
$$\otimes \ : \ \mathrm{Hom}_{\mathbf{C}}(A, A') \times \mathrm{Hom}_{\mathbf{C}}(B, B') \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A \otimes B,\ A' \otimes B'). \tag{109}$$
Since $\mathbf{C}$ is $\mathbb{C}-$linear monoidal, this map is $\mathbb{C}-$bilinear. By the universal property of the tensor product of vector spaces, it therefore induces a canonical $\mathbb{C}-$linear map
$$\Phi_{A,A',B,B'} : \mathrm{Hom}_{\mathbf{C}}(A, A') \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbf{C}}(B, B') \longrightarrow \mathrm{Hom}_{\mathbf{C}}(A \otimes B,\ A' \otimes B'), \tag{110}$$
defined on pure tensors by
$$f \otimes g \longmapsto f \otimes g. \tag{111}$$

In general, the map $\Phi_{A,A',B,B'}$ need not be an isomorphism. When $\Phi_{A,A',B,B'}$ fails to be an isomorphism, we say that $\mathbf{C}$ exhibits a spatial fusion anomaly.

Proposition (Unitary $\mathbb{C}-$linear categories are semisimple) Every unitary $\mathbb{C}-$linear category is semisimple.

Proof Let $\mathbf{C}$ be a unitary $\mathbb{C}-$linear category.

Step 1: Idempotents split Let $A \in \mathrm{ob}(\mathbf{C})$ and let $e : A \to A$ be an idempotent. Consider the morphism
$$p := e^{\dagger} \circ e : A \to A. \tag{112}$$
Then p is self-adjoint and idempotent
$$p^{\dagger} = p, \qquad p^2 = p. \tag{113}$$
In a unitary category, positivity implies that such idempotents correspond to orthogonal projections. Hence A decomposes as
$$A \ \cong \ \mathrm{im}(p) \oplus \ker(p), \tag{114}$$
and e splits. Therefore $\mathbf{C}$ is idempotent complete.

Step 2: Existence of simple objects Let $A \neq 0$ be an object. Consider the finite-dimensional $*-$algebra

$$\mathrm{End}_{\mathbf{C}}(A)$$

with involution given by the dagger. By standard $*-$algebra theory, $\mathrm{End}_{\mathbf{C}}(A)$ contains a minimal nonzero projection p. We shall not prove this step here. By Step 1, p splits, giving a decomposition

$$A \simeq B \oplus C \tag{115}$$

with $B \neq 0$ and $\mathrm{End}_{\mathbf{C}}(B) \simeq \mathbb{C}$. Thus B is simple.

Step 3: Decomposition into simples Applying the above argument iteratively, any object A decomposes as a finite direct sum of simple objects. Orthogonality of projections implies that the resulting simple summands are mutually disjoint.

Conclusion Finite direct sums exist, and every object decomposes as a finite direct sum of mutually disjoint simple objects. Hence $\mathbf{C}$ is semisimple.

Definition (Unitary monoidal category) A unitary monoidal category is a $\mathbb{C}-$linear monoidal category

$$(\mathbf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho) \tag{116}$$

equipped with a unitary structure

$$(-)^\dagger : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{C} \tag{117}$$

such that the monoidal structure is compatible with the dagger. Concretely, the following conditions are required:

1. (Tensor compatibility) For all morphisms $f : A \to A'$ and $g : B \to B'$,

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger. \tag{118}$$

2. (Unit object) The identity morphism on the monoidal unit is unitary:

$$\mathrm{id}_{\mathbb{1}}^\dagger = \mathrm{id}_{\mathbb{1}}. \tag{119}$$

3. (Unitary coherence isomorphisms) The associativity and unit constraints are unitary:

$$\alpha_{A,B,C}^\dagger = \alpha_{A,B,C}^{-1}, \qquad \lambda_A^\dagger = \lambda_A^{-1}, \qquad \rho_A^\dagger = \rho_A^{-1}. \tag{120}$$

Proposition (Left duals give a $\mathbb{C}-$linear functor) Let $\mathbf{C}$ be a $\mathbb{C}-$linear rigid monoidal category. For each object $A \in \mathbf{C}$, choose a left dual A^L together with morphisms

$$\mathrm{coev}_A : \mathbb{1} \to A \otimes A^L, \qquad \mathrm{ev}_A : A^L \otimes A \to \mathbb{1}$$

satisfying the zig-zag identities. Define a map on objects and morphisms by

$$\delta_L : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}, \qquad \delta_L(A) := A^L, \tag{121}$$

and for a morphism $f : A \to B$ define $\delta_L(f) = f^L : B^L \to A^L$ by

$$f^L \equiv \Big(B^L \cong B^L \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{B^L} \otimes \mathrm{coev}_A} B^L \otimes (A \otimes A^L) \cong (B^L \otimes A) \otimes A^L \\ \xrightarrow{(\mathrm{id}_{B^L} \otimes f) \otimes \mathrm{id}_{A^L}} (B^L \otimes B) \otimes A^L \xrightarrow{\mathrm{ev}_B \otimes \mathrm{id}_{A^L}} \mathbb{1} \otimes A^L \cong A^L \Big). \tag{122}$$

Then δ_L is a well-defined functor. Moreover, it is $\mathbb{C}-$linear on hom-spaces.

Proof

(1) Well-defined morphism. The composite 122 has source B^L and target A^L, so indeed $f^L \in \mathrm{Hom}_{\mathbf{C}}(B^L, A^L)$.

(2) Identities. Take $f = \mathrm{id}_A$. Then $(\mathrm{id}_{A^L} \otimes \mathrm{id}_A)$ appears in the middle of 122, so the resulting composite is precisely the standard zig-zag identity showing that the left duality data yields id_{A^L}. Hence

$$(\mathrm{id}_A)^L = \mathrm{id}_{A^L}. \tag{123}$$

(3) Composition (contravariance) Let $f : A \to B$ and $g : B \to C$. Consider $(g \circ f)^L : C^L \to A^L$. Expanding the definition 122, the middle part contains $\mathrm{id}_{C^L} \otimes (g \circ f)$. On the other hand, $f^L \circ g^L$ is the composite

$$C^L \xrightarrow{g^L} B^L \xrightarrow{f^L} A^L, \tag{124}$$

and substituting the defining expressions for g^L and f^L gives a pasting of two duality diagrams. Using associativity/naturality of the tensor product and the zig-zag identities for $(B^L, \mathrm{coev}_B, \mathrm{ev}_B)$, one simplifies this pasting to the single diagram defining $(g \circ f)^L$. Therefore

$$(g \circ f)^L = f^L \circ g^L. \tag{125}$$

This is exactly functoriality for a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{C}$.

(4) **ℂ−linearity** Fix A, B. The assignment

$$\mathrm{Hom}_{\mathbf{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(B^L, A^L), \qquad f \longmapsto f^L$$

is built from tensoring with fixed morphisms (coev_A, ev_B), associativity/unit constraints, and composition. In a $\mathbb{C}$−linear monoidal category, tensoring with a fixed morphism is $\mathbb{C}$−linear and composition is bilinear. Hence for $f, f' : A \to B$ and $\lambda \in \mathbb{C}$,

$$(f + f')^L = f^L + (f')^L, \qquad (\lambda f)^L = \lambda\, f^L. \tag{126}$$

Thus δ_L is $\mathbb{C}$−linear on hom-spaces.

Proposition (Left duals preserve finite direct sums) Let $\mathbf{C}$ be a $\mathbb{C}$−linear rigid monoidal category, and let $A_1, \ldots, A_n \in \mathrm{ob}(\mathbf{C})$ admit a direct sum

$$A \;\cong\; A_1 \oplus \cdots \oplus A_n \tag{127}$$

with structure maps $\iota_i : A_i \to A$ and $\pi_i : A \to A_i$. Then A^L is a direct sum of $A_1^L, \ldots, A_n^L$. Equivalently, there is a canonical isomorphism

$$(A_1 \oplus \cdots \oplus A_n)^L \;\simeq\; A_1^L \oplus \cdots \oplus A_n^L.$$

Proof Apply the previous proposition to the morphisms ι_i and π_i. Since $\delta_L : \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ is a functor, it reverses compositions, so for all i, j,

$$(\pi_i \circ \iota_j)^L = \iota_j^L \circ \pi_i^L.$$

Using the direct sum relations $\pi_i \circ \iota_j = \delta_{ij}\, \mathrm{id}_{A_j}$, we get

$$\iota_j^L \circ \pi_i^L = (\pi_i \circ \iota_j)^L = (\delta_{ij}\, \mathrm{id}_{A_j})^L = \delta_{ij}\, \mathrm{id}_{A_j^L}. \tag{128}$$

Also, from $\sum_{j=1}^n \iota_j \circ \pi_j = \mathrm{id}_A$, functoriality and $\mathbb{C}$−linearity give

$$\sum_{j=1}^n \pi_j^L \circ \iota_j^L = \Big(\sum_{j=1}^n \iota_j \circ \pi_j\Big)^L = (\mathrm{id}_A)^L = \mathrm{id}_{A^L}. \tag{129}$$

Thus the object A^L with morphisms

$$\widetilde{\iota}_i := \pi_i^L : A_i^L \to A^L, \qquad \widetilde{\pi}_i := \iota_i^L : A^L \to A_i^L \tag{130}$$

satisfies the defining equations for a direct sum of $A_1^L, \ldots, A_n^L$.

By uniqueness of direct sums up to unique isomorphism, this exhibits a canonical isomorphism

$$A^L \;\simeq\; \bigoplus_{i=1}^n A_i^L, \tag{131}$$

i.e. $(A_1 \oplus \cdots \oplus A_n)^L \cong A_1^L \oplus \cdots \oplus A_n^L$.

4 More to come

In the next post I'll finally introduce unitary fusion categories and modular tensor categories. This is the final step in this very long mathematical treck towards topological order. Hopefully the post after next I shall finally discuss how all this machinery enters in the description of physical systems, such as the Toric code.